REPORT DOCUMENTATION PAGE

Form Approved
OMB NO. 0704-0188

1. AGENCY USE ONLY (Leave Blank)	2. REPORT DATE see attached	3. REPORT TYPE AND DATES COVERED see memo of transmittal

4. TITLE AND SUBTITLE Nearly Optimal State Feedback Control of Constrained Nonlinear Systems using a Neural Network HJB Approach	5. FUNDING NUMBERS DAAD19-02-1-0366
6. AUTHOR(S) F.L. Lewis	
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) The University of Texas at Arlington Arlington TX 76019-0016	8. PERFORMING ORGANIZATION REPORT NUMBER 26-0401-10
9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES) U. S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211	10. SPONSORING / MONITORING AGENCY REPORT NUMBER -43795-CI .8

11. SUPPLEMENTARY NOTES
The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

12 a. DISTRIBUTION / AVAILABILITY STATEMENT	12 b. DISTRIBUTION CODE
Approved for public release; distribution unlimited.	

13. ABSTRACT (Maximum 200 words)
SEE ATTACHED REPORT

14. SUBJECT TERMS	15. NUMBER OF PAGES
neural network control, nonlinear control, intelligent control	16. PRICE CODE

17. SECURITY CLASSIFICATION OR REPORT	18. SECURITY CLASSIFICATION ON THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL

NSN 7540-01-280-5500
(Rev.2-89)
by ANSI Std. 239-18

Standard Form 298
Prescribed
298-102

20031031 054

Published

~~For submission to~~ IFAC ICONS
Algarve, 8-11 April 2003

NEARLY OPTIMAL STATE FEEDBACK CONTROL OF CONSTRAINED NONLINEAR SYSTEMS USING A NEURAL NETWORK HJB APPROACH

Frank L. Lewis and Murad Abu-Khalaf

Automation and Robotics Research Institute
The University of Texas at Arlington
7300 Jack Newell Blvd. S
Ft. Worth, Texas 76118-7115

Abstract: In this paper, we treat constrained optimization of nonlinear systems. A rigorous solution method to obtain nearly optimal state feedback control that takes into consideration, actuator saturation, state space constraints, and minimum-time control requirement is presented. The constraints are encoded into the optimization formulation through special nonquadrtatic functionals. The associated Hamiton-Jacobi-Bellman (HJB) equation is then solved successively. Nonlinear approximating networks are used to obtain an approximate closed form solution of the value function of the HJB equation, which is then used to obtain a state feedback controller. The solution is carried over a compact set of the asymptotic stability region of an initial stabilizing control.

Keywords: Actuator saturation; Constraints; Minimum-time control; Neural network; Optimal control.

1. INTRODUCTION[1]

Most control systems are required to work under various types of constraints and performance requirements. Many of these constraints can be classified into two main categories. Constraints due to physical limitations on the control input to the plant. A common phenomenon that falls in this category is actuator saturation. Another type of constraints is due to physical limitations of the plant dynamics itself. Examples of these are constraints on the states of the dynamical system itself. An interesting class of optimization problems that arise when having the actuator saturation phenomena is to the minimum-time control problem. Therefore, developing control laws that are optimized for various combinations of system and control constraints and performance requirements comes at the heart of control theory.

The control of systems with saturating actuators has been the focus of many researchers for many years. Several methods for deriving control laws considering the saturation phenomena are found in (Saberi, *et al.*, 1996; Sussmann, *et al.*, 1994). Other methods that deal with constraints on the states of the system as well as the control inputs are found in (Bitsoris and Gravalou, 1999; Henrion, *et al.*, 2001; Gilbert and Tan, 1991). They are based on mathematical programming and the set invariance theory in some cases. The focus in these papers is towards finding stabilizing controllers that are not necessarily in state feedback form, and without considering general optimization issues.

[1] Research supported by grants NSF ECS-0140490 and ARO DAAD 19-02-1-0366

Bernstein (1995) studied the performance optimization of saturated actuators. However, in the work by Lyshevski (1995), (1996), (1998), (2001a), (2001b), a general framework for finding optimal state feedback control is formulated. He proposes the use of nonquadratic performance functionals that enables encoding various kinds of constraints on the control system. These performance functionals are used along with the famous HJB equation that appears in optimal control theory (Lewis and Syrmos, 1995). It is this framework of study that will be the focus of this paper.

For linear system with quadratic performance functionals, the HJB equation becomes the algebraic Riccati equation which is easy to solve. However, when the performance functionals are nonquadratic, or if the dynamics of the system are nonlinear, the resulting HJB equation is highly nonlinear, and is difficult to solve.

Approximate HJB solution has been confronted using many techniques by Saridis and Lee (1979), Beard (1995), Beard, *et al.* (1997), (1998), Murray, *et al.* (2001), Bertsekas and Tsitsiklis (1995), Munos, *et al.* (1999), Kim and Lewis (2000), Han and Balakrishnan (2000), Liu and Balakrishnan (2000), Huang and Lin (1995), and others.

We are interested in closed form solution of the HJB equation which leads to a state feedback control. An interesting solution method of the HJB equation that results in closed form solution is developed by Beard (1995). His solution method is based on a Galerkin approximation of a set of equations, Generalized HJB (GHJB), that appear in the successive approximation theory developed by Saridis and Lee (1979). The successive approximation method improves a given initial stabilizing control. It reduces to the well-known Kleinman (1968) iterative method for solving the algebraic Riccati equation for linear systems.

In this paper, we focus on solving the HJB equation using the successive approximation theory. Since it is developed for the case of quadratic performance functional on the control input, we start first by extending the successive approximation theory to the case of nonquadratic functionals. Then we use the method weighted residuals to get a neural network least squares solutions to the set of successive equations towards obtaining an approximate solution of the HJB equation with nonquadratic performance functionals.

Neural networks have been used to control nonlinear systems, see (Chen and Liu, 1994; Lewis, *et al.* 1999; Polycarpou, 1996; Rovithakis and Christodoulou, 1994; Sadegh, 1993; Sanner and Slotine, 1991). It has been shown that they can effectively extend adaptive control techniques to nonlinearly parameterized systems. The status of neural network control as of 2001 appears in (Narendra and Lewis, 2001).

In (Miller, *et al.*, 1990), Werbos first proposed using neural networks to find optimal control laws using the HJB equation. Parisini (1998) used neural networks to derive optimal control laws for discrete-time stochastic nonlinear system. Therefore, in this paper we employ neural networks to find a nearly optimal solution to the HJB equation for constrained control systems. A preliminary report of this work appears in (Abu-Khalaf and Lewis, 2002).

2. OPTIMAL CONTROL AND THE GENERALIZED HAMILTON-JACOBI BELLMAN EQUATION (GHJB)

Consider an affine in the control nonlinear dynamical system of the form

$$\dot{x} = f(x) + g(x)u(x) \tag{1}$$

where $x \in \Re^n$, $u : \Re^n \to \Re^m$, $f : \Re^n \to \Re^n$, $g : \Re^n \to \Re^n \times \Re^m$. Assume that $f + gu$ is Lipschitz continuous on a set Ω in $\Re^n$ containing the origin, and that the system (1) is controllable in the sense that there exists a continuous control on Ω that asymptotically stabilizes the system.

It is desired to find a control function $u : \Re^n \to \Re^m$, which minimizes a generalized nonquadratic functional

$$V = \int_0^\infty [Q(x) + W(u)]\, dt \tag{2}$$

where $Q(x)$ is positive definite monotonically increasing function on Ω, and thus satisfies the observability condition. $W(u)$ is a positive definite integrand function. For unconstrained control inputs, a common choice for $W(u)$ is

$$W(u) = u^T R u \tag{3}$$

where $R \in \Re^m \times \Re^m$. Note that the control u must not only stabilize the system on Ω, but also make the integral finite. Such controls are defined to be *admissible* (Beard, 1995).

Definition 2.1: *Admissible Controls*

Let $\Psi(\Omega)$ denote the set of admissible controls. A control $u : \Re^n \to \Re^m$ is defined to be admissible with respect to the state penalty function $Q(x)$ on Ω, denoted $u \in \Psi(\Omega)$, if:

- u is continuous on Ω,
- $u(0) = 0$,
- u stabilizes (1) on Ω,
- $\int_0^\infty [Q(x) + W(u)] dt < \infty$, $\forall x \in \Omega$

■

Differentiating V, the value function, along the system trajectories, we obtain what is known as the GHJB equation,

$$
\begin{aligned}
& GHJB(V,u) \stackrel{\Delta}{=} \\
& \frac{\partial V^T}{\partial x}(f+gu)+Q+u^T Ru=0, \qquad (4)\\
& V(0)=0.
\end{aligned}
$$

Note that the GHJB equation becomes the well-known HJB equation on substitution of the optimal control

$$
u^*(x)=-\frac{1}{2}R^{-1}g^T(x)\frac{\partial V^*(x)}{\partial x} \qquad (5)
$$

where $V^*(x)$ is the unique optimal solution to the Hamilton-Jacobi-Bellman (HJB) equation

$$
\begin{aligned}
& HJB(V^*) \stackrel{\Delta}{=} \\
& \frac{\partial V^{*T}}{\partial x}f+Q-\frac{1}{4}\frac{\partial V^{*T}}{\partial x}gR^{-1}g^T\frac{\partial V^*}{\partial x}=0, \qquad (6)\\
& V^*(0)=0.
\end{aligned}
$$

It is shown in (Lyshevski and Meyer, 1995) that the value function obtained from (6) serves as a Lyapunov function on Ω. It is important to note that the GHJB is linear in the value function derivative, while the HJB is nonlinear in the value function derivative. Solving the GHJB requires solving linear partial differential equations, while the HJB equation solution involves nonlinear partial differential equations, which may be impossible to solve. This is the reason for introducing the successive approximation technique using GHJB, which was based on a sound proof in (Saridis and Lee ,1979). In the successive approximation method, one solves (4) for $V(x)$ given a stabilizing control $u(x)$, then finds an improved control based on $V(x)$ using

$$
u=-\frac{1}{2}R^{-1}g^T\frac{\partial V}{\partial x}. \qquad (7)
$$

Saridis and Lee (1979) show that if the initial control $u^{(0)}\in\Psi(\Omega)$, then repetitive application of (4), (7) is a contraction map, and that the sequence of solutions $V^{(i)}$ converges to the optimal HJB solution $V^*(x)$. This of course assumes that someone can solve exactly for (4) at each step.

Although the GHJB equation is in theory easier to solve than the HJB equation, there is no general closed-form solution available to this equation. Beard (1995) used Galerkin's spectral method to get an approximate solution $V(x)$ in (4) at each iteration. He proves convergence in the overall run. See (Beard, *et al.*, 1997, 1998) for complete convergence proofs.

The Galerkin spectral method does not set the GHJB equation to zero at each iteration, but to a residual error instead. It works by placing the solution $V(x)$ to the differential equation in a Hilbert space, and restricting it to a compact set of the stability region of a known initial stabilizing control. It is assumed that we can select a finite basis set $\Phi=\{\phi_j(x)\}_{j=1}^{\infty}$ for $V(x)$, where $\phi_j:\Omega\to\Re$, and $\phi_j(0)=0$ to satisfy the boundary condition $V(0)=0$. It is also assumed that there exist coefficients c_j such that

$$
\|V(x)-V_L(x)\|_{l^2(\Omega)}\to 0, \quad as\ L\to\infty, \qquad (8)
$$

where $V_L(x)=\sum_{j=1}^{L}c_j\phi_j(x)$.

These coefficients are found by setting the projection of the GHJB equation on the finite basis $\Phi=\{\phi_j(x)\}_{j=1}^{\infty}$ to zero $\forall x\in\Omega$; i.e.

$$
\left\langle GHJB\left(\sum_{j=1}^{L}c_j\phi_j;u\right),\phi_n\right\rangle=0, \quad n=1,\cdots,L \qquad (9)
$$

where the inner product is defined as

$$
\langle f,g\rangle\equiv\int_{\Omega}f(x)g(x)\,dx. \qquad (10)
$$

After the coefficients c_j are found, the improved control law becomes,

$$
\begin{aligned}
u_L(x) &= -\frac{1}{2}R^{-1}g^T(x)\frac{dV_L}{dx}(x) \\
&= -\frac{1}{2}R^{-1}g^T(x)\nabla\Phi_L^T C_L
\end{aligned} \qquad (11)
$$

In (Beard, *et al.*, 1997), it is shown that the GHJB converges pointwise and uniformly on Ω as $L\to\infty$. Moreover, (Beard, *et al.*, 1998) shows that $\exists L$ such that the successive approximation theory of (Saridis and Lee ,1979) holds and works with approximate solutions.

The Galerkin approximation technique requires the evaluation of numerous integrals. Moreover, in its current format, the successive approximation algorithm is unable to deal with constrained optimization.

3. SUCCESSIVE APPROXIMATION OF HJB EQUATION WITH CONSTRAINTS ON THE CONTROL SYSTEM

To be able to encode constraints into the HJB equation, complicated nonquadratic performance functionals are required. Moreover, the successive approximation theory must hold for the nonquadratic performance functional used so that it can be employed.

In this section, we will consider three cases which require three types of nonquadratic performance functionals.

3.1 Actuator Saturation

To confront bounds on control inputs of the system, a suitable generalized nonquadratic functional is

$$W(u)=2\int_0^u\left(\phi^{-1}(\mu)\right)^T R d\mu, \tag{12}$$

where $\phi(\cdot):\Re^c \to \Re^m$ is a continuous one-to-one, bounded, real-analytic integrable function of class C^p $(p\geq 1)$ with $\phi(0)=0$, e.g. $\phi(\cdot)\to\tanh(\cdot)$. R is positive definite and assumed to be symmetric for simplicity of analysis. This does not restrict the design criteria on the control input vector, because the number of coefficients that we can choose independently in the symmetric design matrix R is equal to the number of quadratic terms possible from the control input vector. These two numbers are equal, that is $\frac{m^2-m}{2}+m=m+\binom{m}{2}$. Note that $W(u)$ is positive definite if $\phi^{-1}(u)$ has the same sign as u and R is positive definite. Also this functional brings the control signal just to the level of saturation of the actuator, and allows the control signal itself to saturate. Saturation allowance control techniques results in nonlinear closed-loop dynamics, compared to saturation avoidance techniques, (Henrion, *et al.*, 2001).

For saturated controls, GHJB design equations (4), (7) are replaced with

$$\frac{\partial V}{\partial x}^T (f+g\cdot u)+Q+2\int\left(\phi^{-1}(u)\right)^T R du=0, \tag{13}$$

$$V(0)=0.$$

$$u(x)=-\phi\left(\frac{1}{2}R^{-1}g^T(x)\frac{\partial V(x)}{\partial x}\right). \tag{14}$$

Note that equation (14) guarantees that $u(x)$ is bounded.

If we substitute (14) into (13) we obtain the HJB equation for bounded controls. A positive definite solution of this equation is the stabilizing value function and its corresponding optimal control. Existence and uniqueness of the value function has been shown in by Lyshevski (1996). This HJB equation cannot generally be solved. There is no current method for rigorously confronting this type of equation to find the value function of the system. Moreover, current solutions are not well defined over a specific region in the state space.

Successive approximation using the GHJB has not yet been rigorously applied for saturated controls. We will show that the successive approximation technique can be used for constrained controls when certain restrictions on the control input are met. The successive approximation technique is now applied to the new set of equations (13), (14). The following Lemma shows how equation (14) can be used to improve the control law. It will be required that the control bound $\phi(\cdot)$ is monotonically non-decreasing.

<u>Lemma 3.1</u>: *Improved Saturated Control Law*

If $u^{(i)}\in\Psi(\Omega)$, and $V^{(i)}$ satisfies the equation $GHJB(V^{(i)},u^{(i)})=0$ with the boundary condition $V^{(i)}(0)=0$, then the new control derived as

$$u^{(i+1)}(x)=-\phi\left(\frac{1}{2}R^{-1}g^T(x)\frac{\partial V^{(i)}(x)}{\partial x}\right) \tag{15}$$

is an admissible control for the system on Ω. Moreover, if the control bound $\phi(\cdot)$ is monotonically non-decreasing and $V^{(i+1)}$ is the unique positive definite function satisfying the equation $GHJB(V^{(i+1)},u^{(i+1)})=0$, with the boundary condition $V^{(i+1)}(0)=0$, then $V^{(i+1)}(x)\leq V^{(i)}(x)\ \forall x\in\Omega$.

Proof:

Admissibility: Since $V^{(i)}$ is continuously differentiable, the continuity assumption on g implies that $u^{(i+1)}$ is continuous. Since $V^{(i)}$ is positive definite it attains a minimum at the origin, and thus, $\frac{\partial V^{(i)}(x)}{\partial x}$ must vanish. This implies that $u^{(i+1)}(0)=0$.

Taking the derivative of $V^{(i)}$ along the system $(f,g,u^{(i+1)})$ trajectory we have,

$$\dot V^{(i)}(x,u^{(i+1)})=\frac{\partial V^{(i)}(x)}{\partial x}^T f+\frac{\partial V^{(i)}(x)}{\partial x}^T g\ u^{(i+1)}. \tag{16}$$

But,

$$\frac{\partial V^{(i)}(x)}{\partial x}^T f=-\frac{\partial V^{(i)}(x)}{\partial x}^T g\ u^{(i)}-Q(x)- 2\int_0^{u^{(i)}}\left(\phi^{-1}(u)\right)^T R\ du \tag{17}$$

This becomes

$$\dot V^{(i)}(x,u^{(i+1)})=-\frac{\partial V^{(i)}(x)}{\partial x}^T g\ u^{(i)}+\frac{\partial V^{(i)}(x)}{\partial x}^T g\ u^{(i+1)} -Q(x)-2\int_0^{u^{(i)}}\left(\phi^{-1}(u)\right)^T R\ du. \tag{18}$$

Making use of the fact that $\frac{\partial V^{(i)}(x)}{\partial x}^T g(x)=-2\phi^{-1}\left(u^{(i+1)}\right)^T R$, we get

$$\dot V^{(i)}(x,u^{(i+1)})=-Q(x)+ 2\left[\phi^{-1}\left(u^{(i+1)}\right)^T R\ (u^{(i)}-u^{(i+1)})-\int_0^{u^{(i)}}\left(\phi^{-1}(u)\right)^T R\ du\right]. \tag{19}$$

The second term in the previous equation is negative when ϕ^{-1} and thus ϕ is monotonically non-decreasing. To see this, note that the design matrix R is symmetric positive definite, this means we can rewrite it as

$$R=\Lambda\Sigma\Lambda \tag{20}$$

where Σ is a triangular matrix with its values being the singular values of R and Λ is an orthogonal symmetric matrix. Substituting (20) in (19) we get,

$$\dot{V}^{(i)}(x,u^{(i+1)}) = -Q(x) + 2\begin{bmatrix} \phi^{-1}\left(u^{(i+1)}\right)^T \Lambda\Sigma\Lambda\ (u^{(i)} - u^{(i+1)}) - \\ \int_0^{u^{(i)}} \left(\phi^{-1}(u)\right)^T \Lambda\Sigma\Lambda\ du \end{bmatrix}. \tag{21}$$

Applying the coordinate change $u = \Lambda^{-1}z$, equation (21) then becomes

$$\begin{aligned} \dot{V}^{(i)}(x,u^{(i+1)}) &= -Q(x) + 2\begin{Bmatrix} \phi^{-1}\left(\Lambda^{-1}z^{(i+1)}\right)^T \Lambda\Sigma\Lambda\ (\Lambda^{-1}z^{(i)} - \Lambda^{-1}z^{(i+1)}) - \\ \int_0^{z^{(i)}} \left(\phi^{-1}(\Lambda^{-1}z)\right)^T \Lambda\Sigma\Lambda\Lambda^{-1}dz \end{Bmatrix} = \\ &-Q(x) + 2\begin{Bmatrix} \phi^{-1}\left(\Lambda^{-1}z^{(i+1)}\right)^T \Lambda\Sigma(z^{(i)} - z^{(i+1)}) - \\ \int_0^{z^{(i)}} \left(\phi^{-1}(\Lambda^{-1}z)\right)^T \Lambda\Sigma dz \end{Bmatrix} = \\ &-Q(x) + 2\left[\pi^T\left(z^{(i+1)}\right)\Sigma(z^{(i)} - z^{(i+1)}) - \int_0^{z^{(i)}} \pi^T(z)\Sigma dz\right]. \end{aligned} \tag{22}$$

where $\pi^T\left(z^{(i)}\right) = \phi^{-1}\ \left(\Lambda^{-1}z^{(i)}\right)^T \Lambda$.

Since Σ is a triangular matrix, we can now decouple the transformed input vector such that

$$\begin{aligned} \dot{V}^{(i)}(x,u^{(i+1)}) &= -Q(x) + 2\left[\pi^T\left(z^{(i+1)}\right)\Sigma(z^{(i)} - z^{(i+1)}) - \int_0^{z^{(i)}} \pi^T(z)\Sigma dz\right] = \\ &-Q(x) + 2\sum_{k=1}^{m}\Sigma_{kk}\begin{Bmatrix} \pi^T\left(z_k^{(i+1)}\right)(z_k^{(i)} - z_k^{(i+1)}) - \\ \int_0^{z_k^{(i)}} \pi^T(z_k)dz_k \end{Bmatrix}. \end{aligned} \tag{23}$$

Since the matrix R is positive definite, then we have the singular values Σ_{kk} being all positive. Also, from the geometrical meaning of $\pi^T\left(z_k^{(i+1)}\right)(z_k^{(i)} - z_k^{(i+1)}) - \int_0^{z_k^{(i)}} \pi^T(z_k)dz_k$, this term is always negative if $\pi^T(z_k)$ is monotonically non-decreasing. But since $\pi^T\left(z^{(i)}\right) = \phi^{-1}\ \left(\Lambda^{-1}z^{(i)}\right)^T \Lambda$, it is easy to show that $\phi^{-1}(\ )$ should be monotonically non-decreasing, and thus $\phi(\ )$ itself should be monotonically non-decreasing. This implies that $\dot{V}^{(i)}(x,u^{(i+1)}) \le 0$ and that $V^{(i)}(x)$ is a Lyapunov function for $u^{(i+1)}$ on Ω. Following Definition 2.1, $u^{(i+1)}$ is admissible on Ω.

To show the second part of the Lemma 3.1, note that the performance along trajectories $(f, g, u^{(i+1)})$ $\forall x_0$ is,

$$\begin{aligned} &V^{(i+1)} - V^{(i)} = \\ &\int_0^{\infty} Q(x(\tau,x_0,u^{(i+1)})) + \left\|u^{(i+1)}(\tau,x_0,u^{(i+1)})\right\|_R^2 d\tau - \\ &\int_0^{\infty} l(x(\tau,x_0,u^{(i+1)})) + \left\|u^{(i)}(\tau,x_0,u^{(i+1)})\right\|_R^2 d\tau = \\ &-\int_0^{\infty} \frac{d(V^{(i+1)} - V^{(i)})^T}{dx}\left[f + g\,u^{(i+1)}\right]d\tau. \end{aligned} \tag{24}$$

From $GHJB(V^{(i+1)},u^{(i+1)}) = 0$, $GHJB(V^{(i)},u^{(i)}) = 0$, we have

$$\frac{\partial V^{(i)}(x)}{\partial x}^T f = -\frac{\partial V^{(i)}(x)}{\partial x}^T g\ u^{(i)} - l(x) - 2\int_0^{u^{(i)}} \left(\phi^{-1}(u)\right)^T R\ \ du, \tag{25}$$

$$\frac{\partial V^{(i+1)}(x)}{\partial x}^T f = -\frac{\partial V^{(i+1)}(x)}{\partial x}^T g\ u^{(i+1)} - l(x) - 2\int_0^{u^{(i+1)}} \left(\phi^{-1}(u)\right)^T R\ \ du. \tag{26}$$

Substituting (25), (26) in (24) we get

$$V^{(i+1)}(x_0) - V^{(i)}(x_0) = -2\int_0^{\infty}\begin{Bmatrix} \left(\phi^{-1}(u^{(i+1)})\right)^T R(u^{(i+1)} - u^{(i)}) - \\ \int_{u^{(i)}}^{u^{(i+1)}} \left(\phi^{-1}(u)\right)^T R du \end{Bmatrix} d\tau \tag{27}$$

By decoupling the equation (28) using $R = \Lambda\Sigma\Lambda$, it can be shown that

$$V^{(i+1)}(x_0) - V^{(i)}(x_0) \le 0 \tag{29}$$

when $\phi(\cdot)$ is monotonically non-decreasing.

■

The next theorem is a key result on which the rest of the paper is justified. It shows that successive improvement of the saturated control law converges to the optimal saturated control law for the given actuator saturation model $\phi(\cdot)$.

<u>Theorem 3.2</u>: *Convergence of Successive Approximations*

If $u^{(0)} \in \Psi(\Omega)$, then

1. $V^{(i)} \to V^*$ uniformly on Ω
2. $u^{(i)} \in \Psi(\Omega)$, $\forall i \ge 0$
3. $u^{(i)} \to u^*$

Proof: Since $GHJB(V^{(0)},u^{(0)}) = 0$ with appropriate boundary conditions. From Lemma 3.1, we have that $u^{(1)} \in \Psi(\Omega)$ and that $V^{(1)} \le V^{(0)}$. By induction, we have that $V^{(i)} \le V^{(i-1)} \le V^{(0)}$ and $u^{(i)} \in \Psi(\Omega)$. We can repeat the argument used in proof of Lemma 3.1 to show that $V^* \le V^{(i)}$, $\forall i \ge 0$. Thus, $V^{(i)}$ is a monotonically decreasing sequence that is bounded below. Hence $V^{(i)}$ converges to some $V^{(\infty)}$. It is easily then seen that

$$GHJB(V^{(\infty)},u^{(\infty)}) \equiv HJB(V^{(\infty)}) \equiv 0. \tag{30}$$

Then $V^{(\infty)} = V^*$ because of the uniqueness of solutions of the HJB equation (Lewis and Syrmos, 1995, Lyshevski, 1996). And it follows that $u^{(\infty)} = u^*$.

■

The following is a result from (Beard, 1995) which we tailor here to the case of saturated control inputs. It basically guarantees that improving the control law does not reduce the region of asymptotic stability of the initial saturated control law.

Lemma 3.3: *Convergence of Stability Regions*
The saturated optimal control u^* is asymptotically stable on every stability region associated with every improved control law $u^{(i)}$.
Proof: Lemma 3.1 showed that the saturated control $u^{(1)}$ is asymptotically stable on $\Psi^{(0)}$, where $\Psi^{(0)}$ is the stability region of the saturated control $u^{(0)}$. This implies that $\Psi^{(0)} \subseteq \Psi^{(1)}$. From Theorem 3.2, we know that $\Psi^{(i-1)} \subseteq \Psi^{(i)}$ is true. By induction, this implies that

$$\Psi^{(0)} \subseteq \Psi^{(1)} \subseteq \Psi^{(\infty)} \triangleq \Psi^*. \quad (31)$$

■

Lemma 3.4: *Optimal Saturated Control has the Largest Stability Region*
The saturated control u^* has a stability region that is the largest of any other saturated control $u^{(i)}$ that is admissible with respect to $Q(x)$ and the system (f,g).
Proof: Since any admissible saturated control can be thought of as $u^{(0)}$, then from Lemma 3.3, u^* has a stability region that is the largest of any other saturated control that is admissible with respect to $Q(x)$ and the system (f,g).

■

Note that there may be stabilizing saturated controls that have larger stability regions than u^*, but are not admissible with respect to $Q(x)$ and the system (f,g).

3.2 Constrained States

In literature, there exists several techniques that finds a domain of initial states such that starting within this domain guarantees a specific control policy will not violate the constraints, (Gilbert and Tan, 1991). However, we are interested in improving given control laws so that they do not violate specific state space constraints. For this we choose the following nonquadratic performance functional,

$$Q(x,k) = x^T Q x + \sum_{l=1}^{n_c} \left(\frac{x_l}{B_l - \alpha_l} \right)^k \quad (32)$$

where n_c, B_l, are the number of constrained states, the upper bound on x_l respectively. The integer k is an even number, and α_l is a small positive number. As k increases, and $\alpha_l \to 0$, the nonquadratic term will dominate the quadratic term when the state space constraints are violated. However, the nonquadratic term will be dominated by the quadratic term when the state space constraints are not violated. Note that in this approach, the constraints are considered soft constraints that can be hardened by using higher values for k and smaller values for α_l.

3.3 Minimum Time Problems

For systems with saturated actuators, we want to find the control signal required to drive the system to the origin in minimum time. This is done through the following performance functional

$$V = \int_0^\infty \left[\tanh(x^T Q x) + 2\int_0^u \left(\phi^{-1}(\mu)\right)^T R\, d\mu \right] dt. \quad (33)$$

By choosing the coefficients of the weighting matrix R very small, and for $x^T Q x \gg 0$, the performance functional becomes,

$$V = \int_0^{t_s} 1\, dt, \quad (34)$$

and for $x^T Q x \approx 0$, the performance functional becomes,

$$V = \int_{t_s}^\infty \left[x^T Q x + 2\int_0^u \left(\phi^{-1}(\mu)\right)^T R\, d\mu \right] dt. \quad (35)$$

Equation (34) represents usually performance functionals used in minimum-time optimization because the only way to minimize (34) is by minimizing t_s.

Around the time t_s, we have the performance functional slowly switching to a nonquadratic regulator that takes into account the actuator saturation.

Note that this method allows an easy formulation of a minimum-time problem, and that the solution will follow using successive approximation technique. The solution is a nearly minimum-time controller that is easier to find compared with techniques aimed at finding the exact minimum-time controller. Finding an exact minimum-time controller requires finding a bang-bang controller based on a switching surface that is hard to determine (Lewis and Syrmos, 1995; Kirk, 1970).

Having the successive approximation theory well set for nonquadratic functionals, in the next section we will introduce a neural network approximation of the value function, and employ the successive solutions method in a least-squares sense over a compact set of the stability region, Ω. This is far simpler than the Galerkin approximation appearing in (Beard, 1995).

4. NEURAL NETWORK LEAST-SQUARES APPROXIMATE HJB SOLUTION

Although equation (13) is a linear differential equation, when substituting (14) into (13), it is still difficult to solve for the cost function $V^{(i)}(x)$. Therefore, neural networks are now used to approximate the solution for the cost function $V^{(i)}(x)$ at each successive iteration i. Moreover, to approximate integration, a mesh is introduced in $\Re^n$. This yields an efficient, practical, and computationally tractable solution algorithm for general nonlinear systems with states and controls constraints.

4.1 *Neural Network Approximation of the Cost Function* $V(x)$

It is well known that neural networks can be used to approximate smooth functions on prescribed compact sets (Lewis, *et al.* 1999). Since our analysis is restricted to a stability region, which is a compact set, neural networks are natural for our application. Therefore, to successively solve (13), (14) for constrained control systems, we approximate $V^{(i)}(x)$ with a neural network

$$V_L^{(i)}(x)=\sum_{j=1}^{L} w_j^{(i)}\sigma_j(x)=W_L^{T(i)}\bar{\sigma}_L(x), \tag{36}$$

where the activation functions $\sigma_j(x):\Omega\to\Re$, are continuous, $\sigma_j(0)=0$, span $\{\sigma_j\}_1^\infty\subseteq L_2(\Omega)$. The neural network weights are w_j and L is the number of hidden-layer neurons. Vectors $\bar{\sigma}_L(x)\equiv[\sigma_1(x)\ \sigma_2(x)\cdots\sigma_L(x)]^T$, $W_L\equiv[w_1\ w_2\cdots w_L]^T$ are the vector activation function and the vector weight respectively. The neural network weights will be tuned to minimize the residual error in a least-squares sense over a set of points within the stability region of the initial stabilizing control. Least-squares solution attains the lowest possible residual error with respect to the neural network weights.

For the $GHJB(V,u)=0$, the solution V is replaced with V_L having a residual error

$$GHJB\left(V_L(x)=\sum_{j=1}^{L} w_j^{(i)}\sigma_j(x),u\right)=e_L(x). \tag{37}$$

To find the least-squares solution, the method of weighted residuals is used (Finlayson, 1972). The weights w_j are determined by projecting the residual error onto $\dfrac{d\,e_L(x)}{d\,w_L}$ and setting the result to zero $\forall x\in\Omega$, i.e.

$$\left\langle \frac{d\,e_L(x)}{d\,w_L},e_L(x)\right\rangle=0. \tag{38}$$

When expanded, equation (38) becomes,

$$\begin{aligned}&\left\langle\nabla\bar{\sigma}_L(f+gu),\nabla\bar{\sigma}_L(f+gu)\right\rangle w_L+\\&\left\langle Q+2\int\left(\phi^{-1}(u)\right)^T R\,du,\nabla\bar{\sigma}_L(f+gu)\right\rangle=0.\end{aligned} \tag{39}$$

Expanding the derivative of the residual,

$$\begin{aligned}&\left\langle\nabla\bar{\sigma}_L(f+gu),\frac{d\sigma_j}{dx}(f+gu)\right\rangle w_L+\\&\left\langle Q+2\int\left(\phi^{-1}(u)\right)^T R\,du,\frac{d\sigma_j}{dx}(f+gu)\right\rangle=0,\ \ j=1,\cdots,L\end{aligned} \tag{40}$$

The following technical results are needed.

<u>Lemma 4.1</u>: if the set $\{\sigma_j\}_1^L$ is linearly independent and $u\in\Psi(\Omega)$, then the set

$$\left\{\frac{d\sigma_j}{dx}(f+gu)\right\}_1^L \tag{41}$$

is also linearly independent.
Proof: See [3].

■

From Lemma 4.1, equation (40) can be rewritten, after defining $\left\{\frac{d\sigma_j}{dx}(f+gu)\right\}_1^L\triangleq\{\theta_j\}_1^L$, as

$$\begin{aligned}&\left\langle\nabla\bar{\sigma}_L(f+gu),\theta_j\right\rangle w_L+\\&\left\langle Q+2\int\left(\phi^{-1}(u)\right)^T R\,du,\theta_j\right\rangle=0,\quad j=1,\cdots,L.\end{aligned} \tag{42}$$

Because of Lemma 4.1, the term $\left\langle\nabla\bar{\sigma}_L(f+gu),\theta_n\right\rangle$ is of full rank, and thus is invertible. Therefore a unique solution for w_L exists. We can solve equation (42) for w_L as follows,

$$\begin{aligned}w_L=&-\left\langle\nabla\bar{\sigma}_L(f+gu),\theta_j\right\rangle^{-1}\cdot\\&\left\langle Q+2\int\left(\phi^{-1}(u)\right)^T R\,du,\theta_j\right\rangle,\quad j=1,\cdots,L.\end{aligned} \tag{43}$$

4.2 *Introducing a Mesh in* $\Re^n$

Solving the integration in (43) is expensive computationally. However, the integrations can be approximated to a suitable degree using the Riemann definition of integration. This results in a nearly optimal, computationally tractable solution algorithm

<u>Lemma 4.2</u>: *Riemann Approximation of Integrals*
An integral can be approximated as

$$\int_a^b f(x)dx=\lim_{\|\Delta x\|\to 0}\sum_{i=1}^{n} f(\bar{x}_i)\cdot\Delta x, \tag{44}$$

where $\Delta x=x_i-x_{i-1}$ and f is bounded on $[a,b]$, (Burk, 1998).

■

Introducing a mesh on Ω, with mesh size equal to Δx, we can rewrite some terms of (43) as follows:

$$X = \left[\nabla\bar{\sigma}_L(f+gu)\big|_{x1} \cdots \nabla\bar{\sigma}_L(f+gu)\big|_{x_p} \right], \quad (45)$$

$$Y = \begin{bmatrix} Q + 2\int\left(\phi^{-1}(u)\right)^T R du\Big|_{x1} \\ \vdots \\ Q + 2\int\left(\phi^{-1}(u)\right)^T R du\Big|_{x_p} \end{bmatrix}^T, \quad (46)$$

where p in x_p represents the number of points of the mesh. This number increases as the mesh size is reduced.

Using Lemma 4.2, we have

$$\begin{aligned} \left\langle \nabla\bar{\sigma}_L(f+gu), \theta_j \right\rangle &= \lim_{\|\Delta x\|\to 0} \left(X^T X\right)\cdot \Delta x, \\ \left\langle Q + 2\int\left(\phi^{-1}(u)\right)^T R du, \theta_j \right\rangle &= \lim_{\|\Delta x\|\to 0} \left(X^T Y\right)\cdot \Delta x. \end{aligned} \quad (47)$$

This implies that we can calculate w_L as

$$w_L = -\left(X^T X\right)^{-1}\left(X^T Y\right) \quad (48)$$

An interesting observation is that equation (48) is the standard least-squares method of estimation for a mesh on Ω. Note that the mesh size Δ should be such that the number of points p is greater than or equal to the order of approximation L. This guarantees a full rank for $\left(X^T X\right)$.

5. NUMERICAL EXAMPLES

We now show the power of our neural network control technique of finding nearly optimal nonlinear controllers for nonlinear systems. Two examples are presented.

5.1 Nonlinear oscillator with actuator saturation

We consider next a nonlinear oscillator having the dynamics

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2), \\ \dot{x}_2 &= -x_1 + x_2 - x_2(x_1^2 + x_2^2) + u. \end{aligned}$$

It is desired to control the system with control limits of $|u| \le 1$. The following smooth function is used to approximate the value function of the system,

$$\begin{aligned} V_{24}(x_1, x_2) = {} & w_1 x_1^2 + w_2 x_2^2 + w_3 x_1 x_2 + w_4 x_1^4 + w_5 x_2^4 + \\ & w_6 x_1^3 x_2 + w_7 x_1^2 x_2^2 + w_8 x_1 x_2^3 + w_9 x_1^6 + w_{10} x_2^6 \\ & w_{11} x_1^5 x_2 + w_{12} x_1^4 x_2^2 + w_{13} x_1^3 x_2^3 + w_{14} x_1^2 x_2^4 + w_{15} x_1 x_2^5 + \\ & w_{16} x_1^8 + w_{17} x_2^8 + w_{18} x_1^7 x_2 + w_{19} x_1^6 x_2^2 + w_{20} x_1^5 x_2^3 \\ & + w_{21} x_1^4 x_2^4 + w_{22} x_1^3 x_2^5 + w_{23} x_1^2 x_2^6 + w_{24} x_1 x_2^7 \end{aligned}$$

This neural network has 24 activation functions containing powers of the state variable of the system up to the 8^{th} power. The complexity of the neural network is selected to guarantee convergence of the algorithm to an admissible control law. When only up to the 6^{th} order powers are used, convergence of the iteration to admissible controls was not observed.

The state feedback control $u = sat_{-1}^{+1}(-5x_1 - 3x_2)$ is used as an initial stabilizing control for the iteration. This is found after linearizing the nonlinear system around the origin, and building an unconstrained state feedback control which makes the eigenvalues of the linear system all negative. Figure 1 shows the performance of the bounded controller $u = sat_{-1}^{+1}(-5x_1 - 3x_2)$. Note that it is not good.

The nearly optimal saturated control law is now found through the technique presented in this paper.

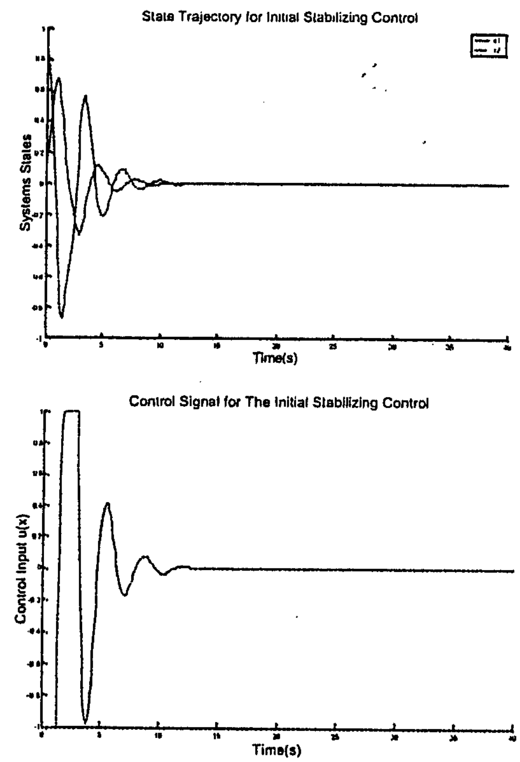

Fig. 1. Performance of the initial stabilizing saturated control.

The algorithm is run over the region $-1 \le x_1 \le 1$, $-1 \le x_2 \le 1$, with a mesh size 0.025, and $R = 1$, $Q = I_{2x2}$. After 20 successive iterations, the nearly optimal saturated control law is found to be,

$$u = -\tanh\begin{pmatrix} 2.62x_1 + 4.23x_2 + 0.39x_2^3 - 4.0x_1^3 - 8.65x_1^2 x_2 \\ -8.94x_1 x_2^2 - 5.53x_2^5 + 2.26x_1^5 + 5.78x_1^4 x_2 + \\ 11.00x_1^3 x_2^2 + 2.57x_1^2 x_2^3 + 2.00x_1 x_2^4 + 2.08x_2^7 \\ -0.49x_1^7 - 1.65x_1^6 x_2 - 2.71x_1^5 x_2^2 - 2.19x_1^4 x_2^3 \\ -0.76x_1^3 x_2^4 + 1.77x_1^2 x_2^5 + 0.87x_1 x_2^6 \end{pmatrix}$$

This is the control law in terms of a neural network. Note that the controller in figure 2 outperforms the initial stabilizing controller shown in figure 1.

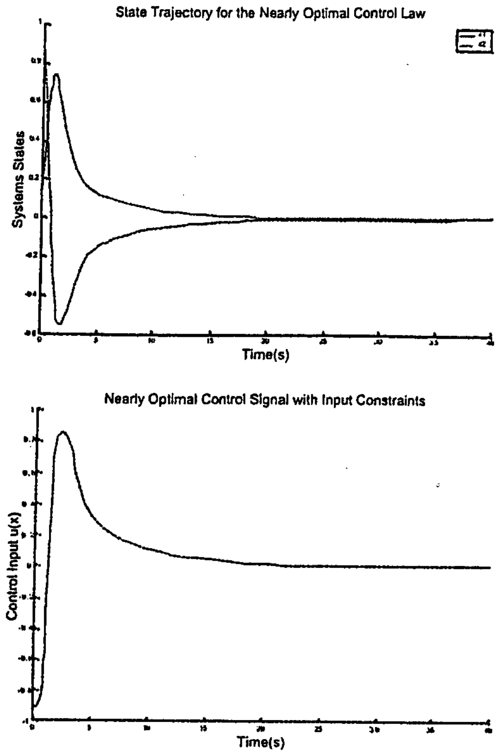

Fig. 2. Nearly optimal nonlinear control law with actuator saturation

5.2 *Constrained state linear system*

Consider the following system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = x_1 + x_2 + u$$
$$|x_1| \le 3.$$

For this we select the following performance functional

$$Q(x,14) = x_1^2 + x_2^2 + \left(\frac{x_1}{3-1}\right)^{10},$$
$$W(u) = u^2.$$

Note that, we have chosen the coefficient k to be 10, and $B_1 = 3$, and $\alpha_1 = 1$. A reason why we have selected k to be 10 is that a larger value for k requires using many activation functions in which a large number of them will have to have powers higher than the value k. However, since this simulation was carried on a double precision computer, then power terms higher than 14 do not add up nicely and round-off errors seriously affect determining the weights of the neural network by causing a rank deficiency.

An initial stabilizing controller, the LQR $-2.4x_1 - 3.6x_2$, that violates the state constraints is shown in figure 3. The performance of this controller is improved by stochastically sampling 3000 times from the region $-3.5 \le x_1 \le 3.5, -5 \le x_2 \le 5$, and running the successive approximation algorithm for 20 times.

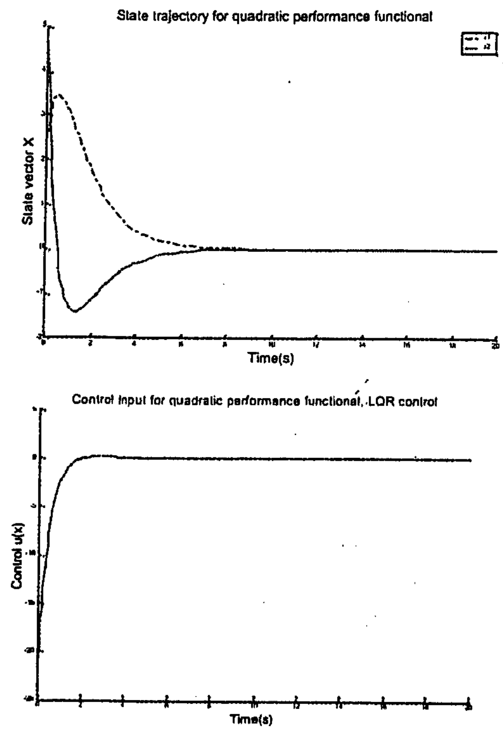

Fig. 3. LQR control without considering the state constraint.

It can be seen that the nearly optimal control law that considers the state constraint tends not to violate the state constraint as the LQR controller does. It is important to realize, that as we increase the order k in the performance functional, then we get larger and larger control signals at the starting time of the control process to avoid violating the state constraints.

A smooth function of the order 45 that resembles the one used in example 5.1 is used to approximate the value function of the system. The weights W_u are found by successive approximation. Since $R = 1$, the final control law becomes,

$$u(x) = -\frac{1}{2} W_u^T \frac{\partial V}{\partial x_2}.$$

It was noted that the nonquadratic performance functional returns an over all cost of 212.33 when the initial conditions are $x_1 = 2.4, x_2 = 5.0$ for the optimal controller, while this cost increases to 316.07 when the linear controller is used. It is this increase in cost detected by the nonquadratic performance functional that causes the system to avoid violating the state constraints. If this difference in costs is made bigger, then we actually increase the set of initial conditions that do not violate the constraint. This however, requires a larger neural network, and high precision computing machines.

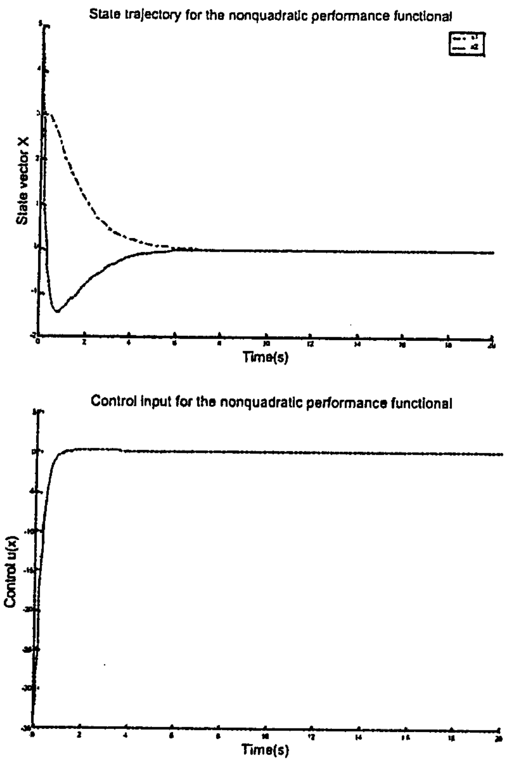

Fig. 4. Nearly optimal nonlinear control law considering

5.3 Minimum time control

Consider the following system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_2 + u.$$

It is desired to control the system with control limits of $|u| \leq 1$ to drive it to origin in minimum time. Typically, from classical optimal control theory, (Kirk 1970), we find out that the control law required is a bang-bang controller that switches back and forth based on a switching surface that is calculated using Pontryagin's minimum principle. It follows that the minimum time control law for this system is given by

$$s(x) = x_1 - \frac{x_2}{|x_2|} \ln\left(|x_2| + 1\right) + x_2,$$

$$u^*(x) = \begin{cases} -1, \text{ for } x \text{ such that } s(x) > 0, \\ +1, \text{ for } x \text{ such that } s(x) < 0, \\ -1, \text{ for } x \text{ such that } s(x) = 0 \text{ and } x_2 < 0, \\ 0, \text{ for } x = 0. \end{cases}$$

The response to this controller is shown in figure 5. It can be seen that this is a highly nonlinear control law, that requires the calculation of a switching surface. This is however a formidable task even for linear systems with state dimension larger than 3. However, when using the method presented in this paper, finding a nearly minimum-time controller becomes a less complicated matter.

We use the following nonquadratic performance functional,

$$Q(x) = \tanh\left(\left(\frac{x_1}{0.1}\right)^2 + \left(\frac{x_2}{0.1}\right)^2\right),$$

$$W(u) = 0.001 * 2\int_0^u \tanh^{-1}(\mu)\, d\mu.$$

A smooth function of the order 35 is used to approximate the value function of the system. We solve for his network by stochastic sampling. By sampling 5000 times from the region $-0.5 \leq x_1 \leq 0.5, -0.5 \leq x_2 \leq 0.5$. The weights W_u are found by successive approximation, for 20 times. Since $R = 1$, the final control law becomes,

$$u(x) = -\tanh\left(\frac{1}{2} W_u^T \frac{\partial V}{\partial x_2}\right).$$

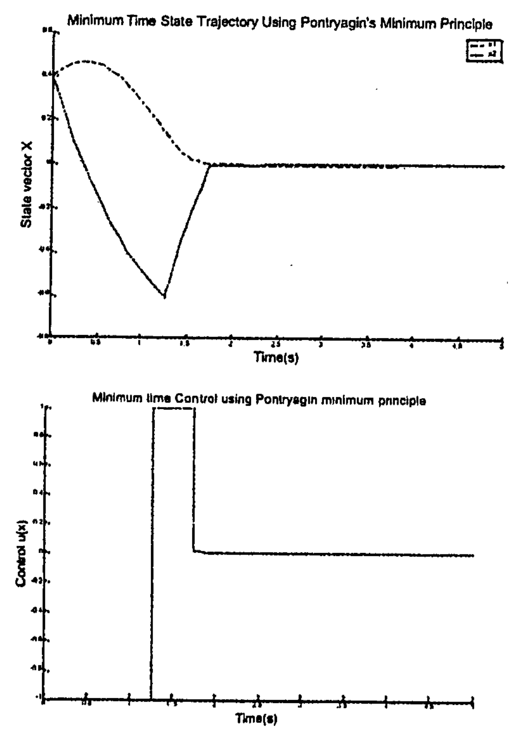

Fig. 5. Performance of the exact minimum-time controller.

Figure 6 shows the performance of the controller obtained using the algorithm presented in this paper. To show how close the performance of this controller to the exact minimum-time controller. Figure 7 plots the state trajectory of both controllers. Note that the nearly minimum-time controller behaves as a bang-bang controller until the states come close to the origin when it starts behaving as a regulator.

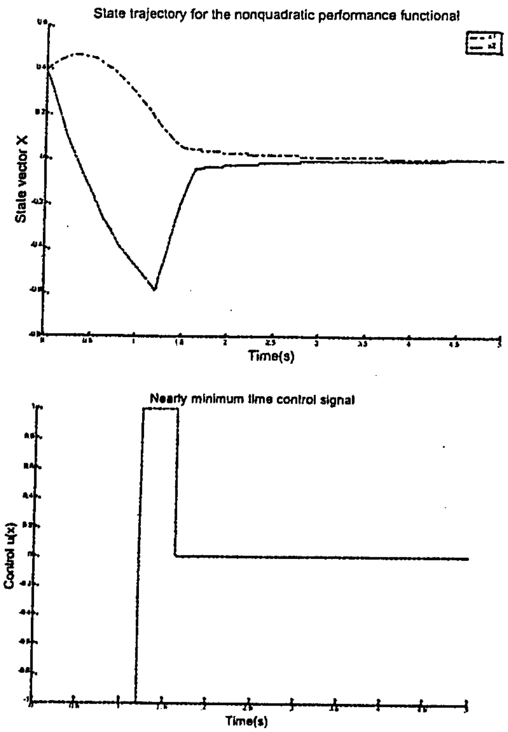

Fig. 6. Performance of the nearly minimum-time controller.

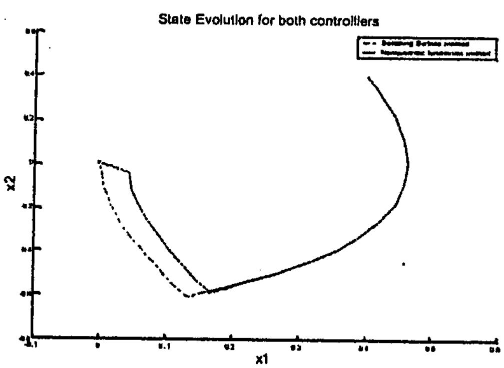

Fig. 7. State evolution for both minimum-time Controllers.

6. CONCLUSION

A rigorous computationally effective algorithm to find nearly optimal control laws under various constraints and requirements is shown. The successive approximation theory has been extended to include nonquadratic performance functionals. The result is a set of control laws that are in state feedback form for general nonlinear systems. The control is given as the output of a neural network. This is an extension of the novel work by Lyshevski (2001) and Beard (1995). Three numerical examples were used to demonstrate the effectiveness of this technique.

REFERENCES

Abu-Khalaf, M., F. L. Lewis (2002). Nearly Optimal HJB Solution for Constrained Input Systems Using a Neural Networks Least-Squares Approach. *Proc. IEEE Conference on Decision and Control*, pp. 541 - 546.

Beard, R. (1995). *Improving the Closed-Loop Performance of Nonlinear Systems*. PhD thesis, Rensselaer Polytechnic Institute, Troy, NY 12180.

Beard, R., G. Saridis, J. Wen (1997). Galerkin Approximations of the Generalized Hamilton-Jacobi-Bellman Equation. *Automatica* 33:12, December, pp. 2159-2177.

Beard, R., G. Saridis, J. Wen (1998). Approximate Solutions to the Time-Invariant Hamilton-Jacobi-Bellman Equation. *Journal of Optimization Theory and Application*, vol 96, no. 3, pp. 589-626.

Bernstein, D. S. (1995). Optimal nonlinear, but continuous, feedback control of systems with saturating actuators. *International Journal of Control*, vol 62, no. 5, pp. 1209-1216.

Bertsekas, D. P., J. N. Tsitsiklis (1996). *Neuro-Dynamic Programming*. Athena Scientific, Belmont, MA.

Bitsoris, G., E. Gravalou. (1999). Design Techniques for the Control of Discrete-Time Systems Subject to State and Control Constraints. *IEEE Trans. Automat. Control*, vol. 44, no. 5, pp. 1057-1061.

Burk, F. (1998). *Lebesgue Measure and Integration*. John Wiley & Sons, New York, NY.

Chen F. C., C.-C. Liu (1994). Adaptively controlling nonlinear continuous-time systems using multilayer neural networks. *IEEE Trans. Automat. Control*, vol. 39, no. 6, pp. 1306-1310.

Finlayson, B. A. (1972). *The Method of Weighted Residuals and Variational Principles*. Academic Press, New York, NY.

Gilbert, E., K. T. Tan (1991). Linear Systems with State and Control Constraints: The Theory and Application of Maximal Output Admissible Sets. *IEEE Trans. Automat. Control*, vol. 36, no. 9, pp. 1008-1020.

Han D., S. N. Balakrishnan (2000). State-Constrained Agile Missile Control with Adaptive-Critic

Based Neural Networks. *Proc. American Control Conference*, pp.1929 – 1933.

Henrion, D., S. Tarbouriech, V. Kucera (2001). Control of Linear Systems Subject to Input Constraints: A polynomial Approach. *Automatica*, vol. 37, no.4, pp.597-604.

Huang, J., C. F. Lin (1995). Numerical Approach to Computing Nonlinear H_∞ Control Laws. *Journal of Guidance, Control, and Dynamics*, vol. 18, no. 5, pp. 989-994.

Khalil, H. (2001). *Nonlinear Systems.* 3rd Edition, Prentice Hall, Upper Saddle River, NJ.

Kim, Y. H., F. L. Lewis, D. Dawson (2000). Intelligent optimal control of robotic manipulators using neural networks. *Automatic 36*, 2000, pp. 1355 – 1364.

Kirk, D. (1970). *Optimal Control Theory: An Introduction.* Prentice Hall, New Jersey.

Kleinman, D. (1968). On an iterative Technique for Riccati Equation Computations. *IEEE Trans. Automatic Control*, pp. 114-115.

Lewis, F. L., V. L. Syrmos (1995). *Optimal Control.* John Wiley & Sons, Inc. New York, NY.

Lewis, F. L., S. Jagannathan, A. Yesildirek (1999). Neural Network Control of Robot Manipulators and Nonlinear Systems. Taylor & Francis.

Liu, X., S. N. Balakrishnan (2000). Convergence Analysis of Adaptive Critic Based Optimal Control. *Proc. American Control Conference*, pp.1929 – 1933.

Lyshevski, S. E., A. U. Meyer (1995). "Control System Analysis and Design Upon the Lyapunov Method," *Proc. American Control Conference*, pp. 3219 - 3223.

Lyshevski, S. E. (1996). Constrained Optimization and Control of Nonlinear Systems: New Results in Optimal Control. *Proc. IEEE Conference on Decision and Control*, pp. 541 - 546.

Lyshevski, S. E. (1998). Optimal Control of Nonlinear Continuous-Time Systems: Design of Bounded Controllers Via Generalized Nonquadratic Functionals. *Proc. American Control Conference*, pp.205 – 209.

Lyshevski, S. E. (2001a). Role of Performance Functionals in Control Laws Design. *Proc. American Control Conference*, pp. 2400 - 2405.

Lyshevski, S. E. (2001b) *Control Systems Theory with Engineering Applications.* Birkhauser, Boston, MA.

Miller, W. T., R. Sutton, P. Werbos (1990). *Neural Networks for Control.* The MIT Press, Cambridge, Massachusetts.

Munos, R. L. C. Baird, A. Moore (1999). Gradient Descent Approaches to Neural-Net-Based Solutions of the Hamilton-Jacobi-Bellman Equation. *International Joint Conference on Neural Networks IJCNN*, vol 3, pp. 2152 – 2157.

Murray, J. C. Cox, R. Saeks, G. Lendaris (2001). Globally Convergent Approximate Dynamic Programming Applied to an Autolander, *Proc. American Control Conference*, pp.2901 - 2906.

Narendra, K. S., F. L. Lewis ed. (2001). Special issue on neural network feedback control. *Automatica* vol 37, no. 8, pp. 1147-1148.

Parisini, T., R. Zoppoli (1998). Neural Approximations for Infinite-Horizon Optimal Control of Nonlinear Stochastic Systems. *IEEE Trans Neural Net.* vol. 9, no. 6, pp. 1388-1408.

Polycarpou, M.M. (1996). Stable adaptive neural control scheme for nonlinear systems. *IEEE Trans. Automat. Control*, vol 41, no. 3, pp. 447-451.

Rovithakis, G.A., M.A. Christodoulou (1994). Adaptive control of unknown plants using dynamical neural networks. *IEEE Trans. Systems, Man, and Cybernetics*, vol 24, no. 3, pp. 400-412.

Saberi, A., Z. Lin, A. Teel (1996). Control of Linear Systems with Saturating Actuators. *IEEE Transactions on Automatic Control*, Vol 41, no. 3, March 1996, pp. 368 -378.

Sadegh, N. (1993). A perceptron network for functional identification and control of nonlinear systems. *IEEE Trans. Neural Networks*, vol. 4, no. 6, pp. 982-988.

Sanner, R.M., J.-J.E. Slotine (1991). Stable adaptive control and recursive identification using radial gaussian networks. *Proc. IEEE Conf. Decision and Control*, pp. 2116-2123.

Saridis, G. C. S. Lee (1979). An Approximation Theory of optimal Control for Trainable Manipulators. *IEEE Trans. Systems, Man, Cybernetics*, vol 9, no. 3, pp. 152-159.

Sussmann, H., E. D. Sontag, Y. Yang (1994). A General Result on the Stabilization of Linear Systems Using Bounded Controls. *IEEE Trans. Automatic Control*, vol 39, no. 12, pp. 2411-2425.